PG – 225

IV Semester M.Sc. Degree Examination, June 2016

(CBCS)

MATHEMATICS

M402TC : Theory of Numbers

Time : 3 Hours Max. Marks : 70

Instructions : 1) Answer any five full questions.

2) All questions carry equal marks.

1. a) Define the Euler totient function $\varphi(n)$. Prove that $\sum_{d|n} \varphi(n) = n$.

   b) Establish a relation connecting the Euler function and the Möbius function.

   c) Prove that $\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$. (5+5+4)

2. a) Prove that the set of all arithmetical functions f with $f(1) \neq 0$ forms an abelian group with respect to Dirichlet multiplication.

   b) Prove that the Möbius function $\mu(n)$ is the sum of the primitive n^{th} roots of unity.

   c) Let f be multiplicative. Then prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$. (6+4+4)

3. a) Define the Bell series $f_p(x)$ of an arithmetical function f modulo p (prime). Obtain the Bell series of the Euler totient function.

   b) State and prove that Selberg identity.

   c) Let f be multiplicative and g be any arithmetical function. Prove that the following statements are equivalent : (3+5+6)

      i) $f(p^{n+1}) = f(p)\, f(p^n) - g(p)\, f(p^{n-1})$ for all primes p and $n \geq 1$.

      ii) $f_p(x) = \frac{1}{1 - f(p)x + g(p)x^2}$.

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4. a) Suppose (a, m) = d and d|b. Then show that the linear congruence $ax \equiv b \pmod{m}$ has exactly d solutions and find them.

   b) For any prime p, prove that all the coefficients of the polynomial $f(x) = (x-1)(x-2) \ldots (x-p+1) - x^{p-1} + 1$ are divisible by p.

   c) For any prime $p \geq 5$, show that $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} \equiv 0 \pmod{p^2}$. (4+4+6)

5. a) State and prove Chinese remainder theorem.

   b) Show that the set of lattice points in the plane visible from the origin contains arbitrarily large square gaps. (7+7)

6. a) Show that the Legendre's symbol (n|p) is a completely multiplicative function of n. Evaluate (2|p).

   b) State and prove Gauss' lemma.

   c) Determine those odd primes p for which 3 is a quadratic residue and those for which it is a non-residue. (4+6+4)

7. a) If $m (\geq 1)$ is not of the form $m = 1, 2, 4, p^{\alpha}$ or $2p^{\alpha}$, where p is an odd prime, then prove that there are no primitive roots mod m.

   b) Prove that the sum of the primitive roots mod p is congruent to $\mu(p-1)$ mod p. (10+4)

8. State and prove Rogers-Ramanujan identities. 14